

UNITARIZABILITY OF WEIGHT MODULES OVER NONCOMMUTATIVE KLEINIAN FIBER PRODUCTS

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ABSTRACT. For any (m, n) -periodic higher spin six-vertex configuration \mathcal{L} , we construct a one-parameter family Δ_ξ of pseudo-unitarizable representations of the corresponding noncommutative fiber product $\mathcal{A}(\mathcal{L})$ by difference operators acting on the space of sections of a complex line bundle L_ξ over the face lattice F . The indefinite inner product is given explicitly in terms of a combinatorial sign function defined on F . We prove that each simple integral weight $\mathcal{A}(\mathcal{L})$ -module (previously classified by the author [3]) occurs as a submodule in one of these representation spaces. Lastly we give a combinatorial description of the signature of the unique (up to nonzero real multiples) indefinite inner product on any simple integral weight module, in terms of certain eight-vertex configurations canonically attached to \mathcal{L} . In particular we obtain necessary and sufficient conditions for such a module to be unitarizable.

1. INTRODUCTION

By an inner product $\langle \cdot, \cdot \rangle$ on a complex vector space V we shall mean in this paper a (not necessarily positive definite) non-degenerate symmetric sesquilinear form. An inner product on a module V over a $*$ -algebra \mathcal{A} is called invariant if $\langle av, w \rangle = \langle v, a^*w \rangle$ for all $a \in \mathcal{A}$ and $v, w \in V$.

An important problem in the representation theory of $*$ -algebras is the question of existence of such forms on modules, and to determine when such a form is positive definite. Modules which can be equipped with an invariant inner product are called pseudo-unitarizable, and unitarizable if the form can be chosen positive definite. The pseudo-unitarizable modules form essentially a “real line” inside the moduli space of all representations (because they are stable under taking the contragredient module). In many cases at most one invariant inner product exists up to equivalence. For example this is the case in general for finite-dimensional indecomposable modules [9].

A classical result states that every complex finite-dimensional representation of (the convolution $*$ -algebra of complex-valued L^1 functions on) a compact topological group G is unitarizable (see e.g. [7, Prop. 4.6]). Other examples from Lie theory include the celebrated discrete series of unitary irreducible highest weight modules over the Virasoro algebra [6], and the classification of pseudo-unitarizable simple weight modules with finite-dimensional weight spaces over a semi-simple complex finite-dimensional Lie algebra with respect to the Chevalley involution [10].

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In this paper we consider a family of $*$ -algebras $\mathcal{A}(\mathcal{L})$ called *noncommutative Kleinian fiber products* [4, 3]. They depend on a certain vertex configuration \mathcal{L} and are noncommutative deformations of the algebra of functions on a fiber product of two type A Kleinian singularities [3]. Examples include central extensions of noncommutative Kleinian singularities introduced by Hodges [5], and quotients of the enveloping algebra of the affine Lie algebra $A_1^{(1)}$ and of the finite W-algebra $\mathcal{W}(\mathfrak{sl}_4, \mathfrak{sl}_2 \oplus \mathfrak{sl}_2)$ [3]. Simple weight $\mathcal{A}(\mathcal{L})$ -modules were classified in [3] and are parametrized by pairs (D, ξ) where D is a connected component of a twisted cylinder minus the edges of \mathcal{L} , and $\xi \in \mathbb{C}$. The algebras $\mathcal{A}(\mathcal{L})$ are examples of rank two twisted generalized Weyl algebras [8]. Pseudo-unitarizable simple and indecomposable weight modules with real support over noncommutative Kleinian singularities, and more generally arbitrary generalized Weyl algebras of rank one, were classified in [2], covering in particular $U_q(\mathfrak{sl}_2)$ at roots of unity q . Bounded and unbounded $*$ -representations of twisted generalized Weyl constructions were studied in [11].

1.1. Summary of paper. In Section 2 we review the definition of noncommutative Kleinian fiber products as given in [3]. In Section 3 we prove the existence of square roots of the polynomial functions $P_i^{\mathcal{L}}$ which still solve the MTE, and use this to construct a one-parameter family Δ_ξ of representations of $\mathcal{A}(\mathcal{L})$. These representations are shown to be pseudo-unitarizable if $|\xi| = 1$ in Section 4. In Section 5 we review the classification of simple integral weight modules from [3] and determine necessary and sufficient conditions for them to be pseudo-unitarizable. Along the way we prove that the Casimir element C for $\mathcal{A}(\mathcal{L})$ defined in [3] is unitary (Section 5.1) and prove a polynomial formula for shifts of products of the square roots of $P_i^{\mathcal{L}}$ (Lemma 5.4). In Section 5.3 we prove that Δ_ξ are completely reducible and that every simple integral weight $\mathcal{A}(\mathcal{L})$ -module occurs as a subrepresentation of Δ_ξ for some ξ . Lastly Section 6 contains the description of the signature of the unique (up to nonzero real multiples) invariant inner product on the simple integral weight modules. In particular we obtain necessary and sufficient conditions for them to be unitarizable. We end with some examples in Section 7.

2. PRELIMINARIES

2.1. Noncommutative Kleinian fiber products. Let (m, n) be a pair of relatively prime non-negative integers, and $(\alpha_1, \alpha_2) = (\alpha, \beta) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ with $m\alpha + n\beta = 0$. Put

$$F = \mathbb{Z}\alpha + \mathbb{Z}\beta \quad V = F + (\alpha + \beta)/2 \quad (2.1)$$

$$E_i = F + \alpha_i/2 \quad E = E_1 \cup E_2 \quad (2.2)$$

Definition 2.1. An (m, n) -periodic higher spin vertex configuration $\mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2)$ is a pair of functions $\mathcal{L}_i : E_i \rightarrow \mathbb{N} = \{0, 1, 2, \dots\}$ with $|\mathcal{L}_i^{-1}([1, \infty))| < \infty$ satisfying the *current conservation rule*:

$$\mathcal{L}_1(v + \beta/2) + \mathcal{L}_2(v + \alpha/2) = \mathcal{L}_1(v - \beta/2) + \mathcal{L}_2(v - \alpha/2) \quad \text{for all } v \in V. \quad (2.3)$$

Let $\tilde{\mathcal{A}} = \tilde{\mathcal{A}}(\mathcal{L})$ be the associative algebra generated by $\{H, X_1^+, X_1^-, X_2^+, X_2^-\}$ subject to defining relations

$$[H, X_i^\pm] = \pm \alpha_i X_i^\pm \quad X_i^\pm X_i^\mp = P_i^\mathcal{L}(H \mp \alpha_i/2) \quad [X_1^\pm, X_2^\mp] = 0 \quad (2.4)$$

where $[a, b] = ab - ba$ and

$$P_i^\mathcal{L}(u) = \prod_{e \in E_i} (u - e)^{\mathcal{L}_i(e)} \quad \text{for } i = 1, 2. \quad (2.5)$$

Let $\mathcal{A} = \mathcal{A}(\mathcal{L}) = \tilde{\mathcal{A}}/\mathcal{I}$ where

$$\mathcal{I} = \{a \in \tilde{\mathcal{A}} \mid p(H)a = 0 \text{ for some nonzero polynomial } p\}. \quad (2.6)$$

Definition 2.2. \mathcal{A} is the *noncommutative Kleinian fiber product associated to \mathcal{L}* .

Note that $(p_1, p_2) = (P_1^\mathcal{L}, P_2^\mathcal{L})$ is a solution to the *Mazorchuk-Turowska Equation (MTE)*

$$p_1(u + \alpha_2/2)p_2(u + \alpha_1/2) = p_1(u - \alpha_2/2)p_2(u - \alpha_1/2) \quad (2.7)$$

which is necessary and sufficient for $\mathcal{A}(\mathcal{L})$ to be nontrivial [3, Prop. 1.11]. Conversely, up to affine transformations any solution (p_1, p_2) to (2.7) is a product of such lattice solutions $(P_1^\mathcal{L}, P_2^\mathcal{L})$ [4, 3].

3. REALIZATION BY DIFFERENCE OPERATORS ON LINE BUNDLES

In this section we construct a natural family of representations Δ_ξ of $\mathcal{A}(\mathcal{L})$ by difference operators acting on global sections of a complex line bundle L_ξ over the one-dimensional lattice F . Later we show that every irreducible integral weight representation of $\mathcal{A}(\mathcal{L})$ is a subrepresentation of Δ_ξ for appropriate ξ . Thus this provides a concrete realization of all simple integral weight modules.

The key result in the construction of Δ_ξ , established in Section 3.1, is the existence of square roots (in fact, logarithms) of solutions to the MTE. The subtlety lies in proving that there exists a consistent choice square root $P_i^\mathcal{L}(e)^{1/2}$ in such a way that the pair of functions still solve the MTE. This choice can be expressed combinatorially directly in terms of the configuration \mathcal{L} (Remark 3.3) and gives rise to both the fundamental symmetry J of the inner product space (Remark 4.8) and formulas for the signature of the inner product given Section 6.

3.1. $\sqrt{\text{MTE}}$. The following shows that solutions to the Mazorchuk-Turowska equation have square roots that are also solutions.

Lemma 3.1. *There exists a pair of functions (q_1, q_2) , $q_i : E_i \rightarrow \mathbb{C}$ such that*

- (i) $[q_i(e)]^2 = P_i^\mathcal{L}(e)$ for all $e \in E_i$ and $i \in \{1, 2\}$,
- (ii) (q_1, q_2) is a solution to the MTE (2.7).

Proof. Put $p_i(u) = P_i^\mathcal{L}(u)$ for $i = 1, 2$. Consider the following pair of functions (l_1, l_2) :

$$l_i(u) = \sum_{e \in E_i, e > u} \mathcal{L}_i(e) \quad \text{for } u \in E_i \text{ and } i = 1, 2. \quad (3.1)$$

Here $e > u$ is the usual order on \mathbb{R} . We claim that these satisfy the following two properties:

$$\frac{p_i(u)}{|p_i(u)|} = (-1)^{l_i(u)} \quad (3.2)$$

and

$$l_1(v + \alpha_2/2) + l_2(v + \alpha_1/2) = l_1(v - \alpha_2/2) + l_2(v - \alpha_1/2) \quad \text{for all } v \in V. \quad (3.3)$$

To check (3.2), use the definition of $p_i(u)$ and that

$$\frac{(u - e)^{\mathcal{L}_i(e)}}{|u - e|^{\mathcal{L}_i(e)}} = \begin{cases} (-1)^{\mathcal{L}_i(e)} & e > u \\ 1 & e < u \end{cases} \quad (3.4)$$

To prove (3.3), substituting (3.1) into (3.3) and cancelling terms we obtain the following, where we assumed WLOG that $\alpha_1 \in \mathbb{Z}_{<0}$ and $\alpha_2 \in \mathbb{Z}_{>0}$:

$$\sum_{\substack{e \in E_1 \\ v - \alpha_2/2 < e \leq v + \alpha_2/2}} \mathcal{L}_1(e) = \sum_{\substack{e \in E_1 \\ v + \alpha_1/2 < e \leq v - \alpha_1/2}} \mathcal{L}_2(e) \quad \text{for all } v \in V. \quad (3.5)$$

Since this equation is additive in \mathcal{L} , we may without loss of generality assume that \mathcal{L} consists of a single generalized Dyck path of period (m, n) . To this end, it is easy to verify (3.5) for the maximum area path consisting of n steps of α_2 followed by m steps of α_1 . An induction argument shows that if (l_1, l_2) solves (3.3) for a certain generalized Dyck path \mathcal{L} , then it also holds for the path in which a 21 step has been replaced with a 12 step. This proves the claim. Now define

$$q_i(e) = \exp(2\pi i \frac{l_i(e)}{4}) |p_i(e)|^{1/2} \quad \text{for all } e \in E_i \text{ and } i = 1, 2. \quad (3.6)$$

where $i^2 = -1$. Then (q_1, q_2) satisfies the required properties. \square

Remark 3.2. Actually the proof shows one can take N :th roots of solutions too:

$$p_i^{1/N}(e) = \exp(2\pi i \frac{l_i(e)}{2N}) |p_i(e)|^{1/N}.$$

Remark 3.3. The combinatorial interpretation of the functions $l_i : E_i \rightarrow \mathbb{N}$ is as follows. For each vertical edge $e \in E_1$, $l_1(e)$ counts the number (with multiplicity) of vertical edges in \mathcal{L} lying above the straight line through e of slope n/m . Similarly for horizontal edges and $l_2(e)$, $e \in E_2$. (The “location” of an edge is by convention its midpoint.) See Figure 1.

3.2. Construction of the representation Δ_ξ . On the discrete space F we define a complex line bundle L_ξ as follows. Fix $\xi \in \mathbb{C}^\times$. Let the abelian group \mathbb{Z} act on $\mathbb{Z}^2 \times \mathbb{C}$ by

$$1.(a, b, z) = (a + m, b + n, \xi z) \quad (3.7)$$

and define $L_\xi = (\mathbb{Z}^2 \times \mathbb{C})/\mathbb{Z}$ with bundle map $L_\xi \rightarrow F$ induced by $(x, y, z) \mapsto x\alpha + y\beta$. Let $\Gamma(L_\xi)$ be the corresponding vector space of global sections. Since $(x, y) \mapsto x\alpha + y\beta$ induces a bijection $\mathbb{Z}^2/\langle(m, n)\rangle \simeq F$, we make the identification

$$\Gamma(L_\xi) = \{f : \mathbb{Z}^2 \rightarrow \mathbb{C} \mid \forall (x, y) \in \mathbb{Z}^2 : f(x - m, y - n) = \xi \cdot f(x, y)\}. \quad (3.8)$$

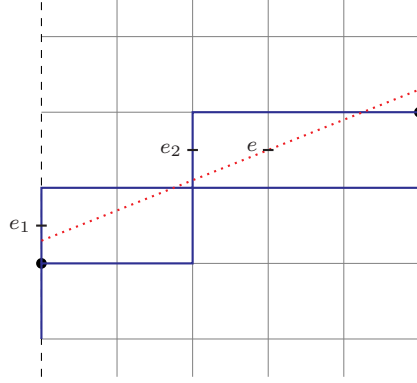


FIGURE 1. A fundamental domain for a $(5, 2)$ -periodic vertex configuration \mathcal{L} (solid blue). Here the vertical edge e has $l_1(e) = 2$ because there are two vertical edges, e_1 and e_2 , appearing in \mathcal{L} above the line (dotted red) through e of slope $5/2$.

It is easy to see that $\Gamma(L_\xi)$ consists of all functions of the form

$$f(x, y) = \tilde{f}(x\alpha + y\beta) \exp\left(\frac{-xm - yn}{m^2 + n^2} \log \xi\right) \quad (3.9)$$

where $\tilde{f} : \mathbb{Z} \rightarrow \mathbb{C}$ is any function and $\log \xi \in \mathbb{C}$ is any choice of logarithm. Consider the subspace

$$\Gamma_0(L_\xi) = \{f \in \Gamma(L_\xi) \mid \tilde{f} \text{ has compact (=finite) support}\}. \quad (3.10)$$

One checks that a \mathbb{C} -basis for $\Gamma_0(L_\xi)$ is given by $\{f_\lambda \mid \lambda \in F\}$ where

$$f_\lambda(x, y) = \delta_{x\alpha+y\beta, \lambda} \exp\left(\frac{-xm - yn}{m^2 + n^2} \log \xi\right) \quad \text{for } \lambda \in F. \quad (3.11)$$

The following theorem shows that the algebra $\mathcal{A}(\mathcal{L})$ acts naturally on $\Gamma_0(L_\xi)$ by difference-multiplication operators.

Theorem 3.4. *For each $\xi \in \mathbb{C}^\times$, there exists a representation*

$$\Delta_\xi : \mathcal{A}(\mathcal{L}) \rightarrow \text{End}_{\mathbb{C}}(\Gamma_0(L_\xi)) \quad (3.12)$$

uniquely determined by

$$(\Delta_\xi(X_1^\pm)f)(x, y) = q_1(x\alpha + y\beta \mp \alpha/2) \cdot f(x \mp 1, y), \quad (3.13a)$$

$$(\Delta_\xi(X_2^\pm)f)(x, y) = q_2(x\alpha + y\beta \mp \beta/2) \cdot f(x, y \mp 1), \quad (3.13b)$$

$$(\Delta_\xi(H)f)(x, y) = (x\alpha + y\beta) \cdot f(x, y), \quad (3.13c)$$

where the q_i where defined in (3.6).

Proof. For brevity, put $\tilde{X}_i^\pm = \Delta_\xi(X_i^\pm)$, $\tilde{H} = \Delta_\xi(H)$ and $p_i = P_i^\mathcal{L}$. We have

$$\begin{aligned} (\tilde{X}_1^\pm \tilde{X}_1^\mp f)(x, y) &= q_1(x\alpha + y\beta \mp \alpha/2) \cdot (\tilde{X}_1^\mp f)(x \mp 1, y) \\ &= q_1(x\alpha + y\beta \mp \alpha/2) q_1((x \mp 1)\alpha + y\beta \pm \alpha/2) \cdot f(x, y) \\ &= p_1(x\alpha + y\beta \mp \alpha/2) \cdot f(x, y) \\ &= (p_1(\tilde{H} \mp \alpha/2)f)(x, y). \end{aligned}$$

This shows that $\tilde{X}_1^\pm \tilde{X}_1^\mp = p_1(\tilde{H} \mp \alpha/2)$. Similarly one checks that $\tilde{X}_2^\pm \tilde{X}_2^\mp = p_2(\tilde{H} \mp \beta/2)$.

Next we verify that $[\tilde{H}, \tilde{X}_i^\pm] = \pm \alpha_i \tilde{X}_i^\pm$ where $(\alpha_1, \alpha_2) = (\alpha, \beta)$ for brevity. We have

$$\begin{aligned} ([\tilde{H}, \tilde{X}_1^\pm]f)(x, y) &= (x\alpha + y\beta) \cdot (\tilde{X}_1^\pm f)(x, y) - q_1(x\alpha + y\beta \mp \alpha/2)(\tilde{H}f)(x \mp 1, y) \\ &= (x\alpha + y\beta) q_1(x\alpha + y\beta \mp \alpha/2) f(x \mp 1, y) \\ &\quad - q_1(x\alpha + y\beta \mp \alpha/2)((x \mp 1)\alpha + y\beta) f(x \mp 1, y) \\ &= \pm \alpha q_1(x\alpha + y\beta \mp \alpha/2) f(x \mp 1, y) \\ &= (\pm \alpha \tilde{X}_1^\pm f)(x, y) \end{aligned}$$

and similarly for \tilde{X}_2^\pm .

Next, the most crucial calculation is to verify that $[\tilde{X}_1^\pm, \tilde{X}_2^\mp] = 0$ which is where we need that (q_1, q_2) satisfy the MTE (2.7).

$$\begin{aligned} ([\tilde{X}_1^+, \tilde{X}_2^-]f)(x, y) &= (\tilde{X}_1^+ \tilde{X}_2^- f)(x, y) - (\tilde{X}_2^- \tilde{X}_1^+ f)(x, y) \\ &= q_1(x\alpha + y\beta - \alpha/2)(\tilde{X}_2^- f)(x - 1, y) - q_2(x\alpha + y\beta + \beta/2)(\tilde{X}_1^+ f)(x, y + 1) \\ &= q_1(x\alpha + y\beta - \alpha/2) q_2(x\alpha + y\beta - \alpha + \beta/2) f(x - 1, y + 1) \\ &\quad - q_1(x\alpha + y\beta - \alpha/2 + \beta) q_2(x\alpha + y\beta + \beta/2) f(x - 1, y + 1) \\ &= (q_1(v - \beta/2) q_2(v - \alpha/2) - q_1(v + \beta/2) q_2(v + \alpha/2)) f(x + 1, y - 1) \\ &= 0 \end{aligned}$$

where we put $v = x\alpha + y\beta - \alpha/2 + \beta/2$. By $1 \leftrightarrow 2$ the other case also holds. This shows that (3.13) defines a homomorphism $\Delta_\xi : \tilde{\mathcal{A}}(\mathcal{L}) \rightarrow \text{End}_{\mathbb{C}}(\Gamma_0(L_\xi))$.

It remains to show that the torsion ideal \mathcal{J} in (2.6) is in the kernel of Δ_ξ . Since \mathcal{J} is a graded ideal with respect to the \mathbb{Z}^2 -gradation on $\tilde{\mathcal{A}}(\mathcal{L})$ given by $\deg X_i^\pm = \pm e_i$, $\deg H = 0$, this amounts to proving that if $d = (d_1, d_2) \in \mathbb{Z}^2$ and $a \in \tilde{\mathcal{A}}(\mathcal{L})_d$ belongs to \mathcal{J} , then $\Delta_\xi(a) = 0$. Since $a \in \mathcal{J}$ there exists a nonzero polynomial g such that $g(H) \cdot a = 0$ in $\tilde{\mathcal{A}}(\mathcal{L})$. Applying Δ_ξ we obtain

$$g(\tilde{H}) \cdot \Delta_\xi(a) = 0. \quad (3.14)$$

In (3.14), acting on an arbitrary $f \in \Gamma_0(L_\xi)$ gives

$$g(x\alpha + y\beta) \cdot (\Delta_\xi(a)f)(x, y) = 0. \quad (3.15)$$

By (3.13) there exists a function $h : F \rightarrow \mathbb{C}$ such that

$$(\Delta_\xi(a)f)(x, y) = h(x\alpha + y\beta) f(x + d_1, y + d_2) \quad (3.16)$$

hence

$$g(x\alpha + y\beta)h(x\alpha + y\beta)f(x + d_1, y + d_2) = 0. \quad (3.17)$$

Choosing f as the basis vectors f_λ defined in (3.11), we obtain

$$g(\lambda)h(\lambda) = 0 \quad \text{for all } \lambda \in F. \quad (3.18)$$

By (3.6) and (3.13), $h(\lambda)$ given in (3.16) is real analytic in a region $\lambda > N$ for $N \gg 0$ while g is a non-zero polynomial, so (3.18) implies that h is identically zero. This shows that $\Delta_\xi(a) = 0$. This completes the proof of the existence of the homomorphism Δ_ξ . The uniqueness follows from the fact that $\mathcal{A}(\mathcal{L})$ is generated by the elements X_i^\pm and H . \square

4. PSEUDO-UNITARIZABILITY OF Δ_ξ

In Section 4.1 we review the basic definitions needed for the following section, where we prove that Δ_ξ is pseudo-unitarizable when $|\xi| = 1$.

4.1. Pseudo-unitarizable modules over \ast -algebras. In this subsection let \mathcal{A} denote a \ast -algebra, by which we mean an associative unital algebra over \mathbb{C} equipped with a conjugate-linear map $\mathcal{A} \rightarrow \mathcal{A}, a \mapsto a^\ast$ satisfying

$$(ab)^\ast = b^\ast a^\ast \quad (a^\ast)^\ast = a \quad \text{for all } a, b \in \mathcal{A}. \quad (4.1)$$

Definition 4.1. Let M be a module over \mathcal{A} . By an *inner product* on M ,

$$\langle \cdot, \cdot \rangle : M \times M \rightarrow \mathbb{C}$$

we mean a non-degenerate symmetric sesquilinear form:

- (i) $\langle \lambda u + \mu v, w \rangle = \lambda \langle u, w \rangle + \mu \langle v, w \rangle$ for all $u, v, w \in M$ and $\lambda, \mu \in \mathbb{C}$,
- (ii) $\langle v, w \rangle = \overline{\langle w, v \rangle}$ for all $v, w \in M$, where the bar denotes complex conjugation,
- (iii) if $\langle v, w \rangle = 0$ for all $v \in M$ then $w = 0$.

An inner product $\langle \cdot, \cdot \rangle$ on M is called *(\ast)-invariant* if

- (iv) $\langle av, w \rangle = \langle v, a^\ast w \rangle$ for all $a \in \mathcal{A}, v, w \in M$

and *positive definite* if

- (v) $\langle v, v \rangle > 0$ for all nonzero $v \in V$.

Definition 4.2. Let M be an \mathcal{A} -module. Then M is *pseudo-unitarizable* if there exists an invariant inner product on M and *unitarizable* if there exists a positive definite invariant inner product on M .

Definition 4.3. The *finitistic dual* of an \mathcal{A} -module with a decomposition $M = \bigoplus_{\lambda \in \mathbb{C}} M_\lambda$, $\dim_{\mathbb{C}} M_\lambda < \infty$, is defined as

$$M^\# = \bigoplus_{\lambda \in \mathbb{C}} M_\lambda^\#, \quad M_\lambda^\# = \{f : M_\lambda \rightarrow \mathbb{C} \mid f \text{ is conjugate-linear}\}$$

with \mathcal{A} -action

$$(af)(v) = f(a^\ast v), \quad \forall a \in \mathcal{A}, f \in M_\lambda^\#, v \in M_\lambda, \lambda \in \mathbb{C}.$$

Theorem 4.4. (a) M is pseudo-unitarizable if and only if $M^\# \simeq M$.

- (b) *If M is indecomposable there is at most one invariant inner product on M , up to equivalence.*

Proof. Follows from general results in [9]. \square

4.2. Pseudo-unitarizability of Δ_ξ . The algebras $\mathcal{A}(\mathcal{L})$ become $*$ -algebras by defining

$$H^* = H \quad (X_i^\pm)^* = X_i^\mp \quad \text{for } i = 1, 2. \quad (4.2)$$

In this section we give an explicit invariant inner product on the representation space $\Gamma_0(L_\xi)$. We assume $|\xi| = 1$ and write $\xi = e^{2\pi i \kappa}$. Every $f \in \Gamma_0(L_\xi)$ has the form

$$f(x, y) = \tilde{f}(x\alpha + y\beta) \exp\left(2\pi i \frac{-xm - yn}{m^2 + n^2} \kappa\right) \quad (4.3)$$

for some unique function $\tilde{f} : F \rightarrow \mathbb{C}$ of finite support.

Theorem 4.5. Δ_ξ is pseudo-unitarizable. More precisely, there exists a weight function $w : F \rightarrow \{+1, -1\}$ such that

$$\langle f, g \rangle = \sum_{\lambda \in F} \tilde{f}(\lambda) \overline{\tilde{g}(\lambda)} w(\lambda) \quad (4.4)$$

is a binary form on $\Gamma_0(L_\xi)$ satisfying (i)–(iv) of Definition 4.2.

Proof. The form is invariant iff $\langle \Delta_\xi(X_i^\pm) f, g \rangle = \langle f, \Delta_\xi(X_i^\mp) g \rangle$ because $\mathcal{A}(\mathcal{L})$ is generated by X_i^\pm and H , and that $\langle \Delta_\xi(H) f, g \rangle = \langle f, \Delta_\xi(H) g \rangle$ holds is immediate because $\Delta_\xi(H)$ is diagonal with real eigenvalues. We have

$$\begin{aligned} \langle \Delta_\xi(X_1^\pm) f, g \rangle &= \sum_{\lambda \in F} q_1(\lambda \mp \frac{\alpha}{2}) \tilde{f}(\lambda \mp \alpha) \overline{\tilde{g}(\lambda)} w(\lambda) \\ &= \sum_{\lambda \in F} q_1(\lambda \pm \frac{\alpha}{2}) \tilde{f}(\lambda) \overline{\tilde{g}(\lambda \pm \alpha)} w(\lambda \pm \alpha) \\ &= \sum_{\lambda \in F} \tilde{f}(\lambda) \overline{q_1(\lambda \pm \frac{\alpha}{2}) \tilde{g}(\lambda \pm \alpha)} \left(\frac{q_1(\lambda \pm \frac{\alpha}{2})}{|q_1(\lambda \pm \frac{\alpha}{2})|} \right)^2 w(\lambda \pm \alpha) \end{aligned}$$

and similarly for X_2^\pm and β which leads to the conditions

$$\left(\frac{q_i(\lambda \pm \frac{\alpha_i}{2})}{|q_i(\lambda \pm \frac{\alpha_i}{2})|} \right)^2 w(\lambda \pm \alpha_i) = w(\lambda) \quad \text{for } i = 1, 2. \quad (4.5)$$

If (4.5) hold then the form $\langle \cdot, \cdot \rangle$ defined by (4.4) is invariant. Substituting (3.6) into (4.5) we obtain

$$\exp(2\pi i l_i(\lambda \pm \frac{\alpha_i}{2})/2) w(\lambda \pm \alpha_i) = w(\lambda), \quad i = 1, 2 \quad (4.6)$$

Again it suffices to take a phase function

$$w(\lambda) = \exp(2\pi i \omega(\lambda)) \quad (4.7)$$

The system of difference equations for $\omega(\lambda)$ can then be written

$$\omega(\lambda \pm \alpha_i) \equiv_{\mathbb{Z}} \omega(\lambda) + \frac{1}{2}l_i(\lambda \pm \frac{\alpha_i}{2}), \quad i = 1, 2 \quad (4.8)$$

where $a \equiv_{\mathbb{Z}} b$ iff $a - b \in \mathbb{Z}$. For this system to have solutions the l_i must satisfy consistency equations which can be written

$$\frac{1}{2}l_1(\lambda - \frac{\alpha_2}{2}) + \frac{1}{2}l_2(\lambda - \frac{\alpha_1}{2}) \equiv_{\mathbb{Z}} \frac{1}{2}l_1(\lambda + \frac{\alpha_2}{2}) + \frac{1}{2}l_2(\lambda + \frac{\alpha_1}{2}) \quad (4.9)$$

which actually holds as an equality due to the current conservation (3.3). This proves that the system of difference equations is consistent and with boundary condition $\omega(0) = 0$ we obtain the unique solution

$$\omega(\pm(\alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_k})) = \frac{1}{2} \sum_{r=1}^k l_{i_r} \left(\pm(\alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_{r-1}} + \frac{\alpha_{i_r}}{2}) \right) \quad (4.10)$$

for any sequence $\underline{i} = i_1 i_2 \dots i_k \in \text{Seq}_2$. Since $\mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2 = F$ and $m\alpha_1 + n\alpha_2 = 0$ there are non-negative integers a, b such that $F = \langle a\alpha_1 + b\alpha_2 \rangle$ as abelian groups, proving that the elements $\pm(\alpha_{i_1} + \cdots + \alpha_{i_k})$ run through all of F . Due to (4.9) the value of ω modulo \mathbb{Z} is independent of \underline{i} . This gives a unique solution $w(\lambda)$ to (4.6) having $w(0) = 1$.

That this form is symmetric $\langle f, g \rangle = \overline{\langle g, f \rangle}$ follows from the fact that $w(\lambda)$ is real-valued. Actually $w(\lambda) \in \{1, -1\}$ for all $\lambda \in F$ because $l_i(\lambda)$ are integer valued hence $\omega(\lambda) \in \frac{1}{2}\mathbb{Z}$.

Finally $\langle \cdot, \cdot \rangle$ is non-degenerate: If $\langle f, g \rangle = 0$ for all g , we can pick $g = f_\lambda$, see (3.11). Then $\tilde{f}_\lambda(\mu) = \delta_{\lambda, \mu}$ where δ is Kronecker's delta, hence

$$\langle f, f_\lambda \rangle = \pm \tilde{f}(\lambda), \quad \forall \lambda \in F$$

hence $\tilde{f}(\lambda) = 0$ for all $\lambda \in F$ which implies that f is identically zero. \square

Remark 4.6. This means we have produced an explicit isomorphism $\Gamma_0(L_\xi) \cong \Gamma_0(L_\xi)^\#$, namely $f \mapsto \langle f, \cdot \rangle$.

Remark 4.7. When $|\xi| = 1$, this gives the following independent proof that $\Delta_\xi(\mathcal{J}) = 0$. Since \mathcal{J} is a graded ideal with respect to the \mathbb{Z}^2 -grading on $\mathcal{A}(\mathcal{L})$ it suffices to prove that $\mathcal{J}_d \in \ker \Delta_\xi$ for each $d \in \mathbb{Z}^2$. Let $a \in \mathcal{J}_d$. Then $a^*a = 0$ by [3, Thm. 3.11(ii) \Rightarrow (i)]. Thus we have for any $\lambda \in F$,

$$\langle \Delta_\xi(a)f_\lambda, \Delta_\xi(a)f_\lambda \rangle = \langle \Delta_\xi(a^*a)f_\lambda, f_\lambda \rangle = 0.$$

Since the form is non-degenerate, and all weight spaces are one-dimensional and pairwise orthogonal, there are no nonzero isotropic weight vectors. This implies that $\Delta_\xi(a) = 0$.

Remark 4.8. In the language of [1], the *fundamental symmetry* operator for the (indefinite) inner product space $\Gamma_0(L_\xi)$,

$$J : \Gamma_0(L_\xi) \rightarrow \Gamma_0(L_\xi)$$

is given by

$$Jf_\lambda = w(\lambda)f_\lambda \quad \text{for all } \lambda \in F.$$

and the J -eigenspace decomposition of $\Gamma_0(L_\xi)$

$$\Gamma_0(L_\xi) = \Gamma_0(L_\xi)^+ \oplus \Gamma_0(L_\xi)^-$$

is the *fundamental decomposition*. On the $+1$ (respectively -1) eigenspace the form $\langle \cdot, \cdot \rangle$ is positive (respectively negative) definite.

5. RELATION TO SIMPLE INTEGRAL WEIGHT MODULES

In this section we prove that the representations Δ_ξ are completely reducible. Moreover, every simple integral weight module occurs as a subspace in $\Gamma_0(L_\xi)$ for some ξ .

5.1. Unitarity of the Casimir. In [3] an $\mathcal{A}(\mathcal{L})$ -centralizing element of the localization $\mathcal{A}(\mathcal{L})_{\text{loc}} = \mathcal{A}(\mathcal{L}) = \otimes_{\mathbb{C}[H]} \mathbb{C}(H)$ was given.

Theorem 5.1 ([3, Prp. 6.3, Thm. C]). *Consider the element $C \in \mathcal{A}(\mathcal{L})_{\text{loc}}$ given by*

$$C = X(\underline{i}) \prod_{\lambda \in F} (H - \lambda)^{-\text{ord}(\underline{i}, \lambda)} \quad (5.1)$$

where $\underline{i} = i_1 i_2 \dots i_{m+n} \in \text{Seq}_2(m, n)$ is a sequence of m 1's and n 2's in any order, $X(\underline{i}) = X_{i_{m+n}}^+ \dots X_{i_2}^+ X_{i_1}^+$, and $\text{ord}(\underline{i}, \lambda)$ is the number (with multiplicity) of vertical (equivalently, horizontal) edges in \mathcal{L} intersected by the face lattice path $\lambda, \lambda + \alpha_{i_1}, \lambda + \alpha_{i_1} + \alpha_{i_2}, \dots, \lambda + \alpha_{i_1} + \alpha_{i_2} + \dots + \alpha_{i_{m+n}} = \lambda$.

- (a) C is independent of the choice of \underline{i} from $\text{Seq}_2(m, n)$.
- (b) C belongs to the center of $\mathcal{A}(\mathcal{L})_{\text{loc}}$. In particular, $[C, \mathcal{A}(\mathcal{L})] = 0$.
- (c) $C \in \mathcal{A}(\mathcal{L})$ iff \mathcal{L} is a five-vertex configuration (every vertex has at most two incident edges with nonzero multiplicity) in which case $Z(\mathcal{A}(\mathcal{L})) = \mathbb{C}[C, C^{-1}]$. Otherwise $Z(\mathcal{A}(\mathcal{L})) = \mathbb{C}$.

Definition 5.2. We call the element C the *Casimir element* for $\mathcal{A}(\mathcal{L})$.

In [3] it was observed that C^*C is a constant (C^*C has \mathbb{Z}^2 -degree $(0, 0)$ and thus is a polynomial in H , but it also commutes with X_i^\pm and is therefore a constant, a priori some rational number). In this subsection we prove that in fact this number equals 1, i.e. C is unitary. This further shows that C is some canonical object.

First we prove two lemmas.

Lemma 5.3. *For any $\lambda \in F$ and $\underline{i} \in \text{Seq}_2(m, n)$ we have (putting $\ell = \ell(\underline{i}) = m + n$),*

$$\sum_{j=1}^{\ell} \mathcal{L}_{i_j}(\lambda + \alpha_{i_1} + \alpha_{i_2} + \dots + \alpha_{i_{j-1}} + \frac{\alpha_{i_j}}{2}) = 2 \text{ord}(\underline{i}, \lambda) \quad (5.2)$$

Proof. The left hand side counts the total number of both vertical and horizontal edges in \mathcal{L} (with multiplicity) that the face path (\underline{i}, λ) intersects. Since \underline{i} is a loop, the number of horizontal edges it intersect from \mathcal{L} is the same as the number of vertical edges it crosses, and this number is exactly equal to the order of the path (\underline{i}, λ) , see [3, Lem. 5.11]. \square

Put

$$q_{\underline{i}}(H) = q_{i_1}(H + \frac{\alpha_{i_1}}{2})q_{i_2}(H + \alpha_{i_1} + \frac{\alpha_{i_2}}{2}) \cdots q_{i_\ell}(H + \alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_{\ell-1}} + \frac{\alpha_{i_\ell}}{2}) \quad (5.3)$$

and similarly for $p_{\underline{i}}(H)$. Remarkably, even though each $q_j(H)$ defined in (3.6) is locally a branch of the square root of the polynomial $P_j^{\mathcal{L}}(H)$, the following lemma shows that $q_{\underline{i}}(H)$ is a polynomial, provided the sequence \underline{i} consists of m 1's and n 2's in any order.

Lemma 5.4. *For any $\underline{i} \in \text{Seq}_2(m, n)$ we have, as functions on F ,*

$$q_{\underline{i}}(H) = \prod_{\lambda \in F} (H - \lambda)^{\text{ord}(\underline{i}, \lambda)}. \quad (5.4)$$

Proof. Put $p_i(H) = P_i^{\mathcal{L}}(H)$ and $\ell = \ell(\underline{i}) = m + n$. Then

$$(q_{\underline{i}}(H))^2 = \prod_{j=1}^{\ell} p_{i_j}(H + \alpha_{i_1} + \cdots + \alpha_{i_{j-1}} + \frac{\alpha_{i_j}}{2}). \quad (5.5)$$

Now note that

$$p_k(H + \frac{\alpha_k}{2}) = \prod_{e \in E_k} (H + \frac{\alpha_k}{2} - e)^{\mathcal{L}_k(e)} \quad (5.6)$$

since $E_k = F + \alpha_k/2$ we make the substitution $\lambda = e - \alpha_k/2$ to rewrite (5.6) as

$$p_k(H + \frac{\alpha_k}{2}) = \prod_{\lambda \in F} (H - \lambda)^{\mathcal{L}_k(\lambda + \frac{\alpha_k}{2})} \quad (5.7)$$

Applying (5.7) to each factor in (5.5) we get

$$\begin{aligned} p_{\underline{i}}(H) &= \prod_{j=1}^{\ell} \prod_{\lambda \in F} (H + \alpha_{i_1} + \cdots + \alpha_{i_{j-1}} - \lambda)^{\mathcal{L}_{i_j}(\lambda + \frac{\alpha_{i_j}}{2})} \\ &= \prod_{\lambda \in F} (H - \lambda)^{\sum_{j=1}^{\ell} \mathcal{L}_{i_j}(\lambda + \alpha_{i_1} + \cdots + \alpha_{i_{j-1}} + \frac{\alpha_{i_j}}{2})} \\ &= \prod_{\lambda \in F} (H - \lambda)^{2 \text{ord}(\underline{i}, \lambda)} \end{aligned}$$

where we used Lemma 5.3 in the last step.

It remains to show both sides have the same sign. When evaluating at $H = \mu \in F$, both sides of (5.4) have the same zero set, namely the set of all $\mu \in F$ such that $\text{ord}(\underline{i}, \mu) > 0$. So it suffices to show that both sides have the same sign when they are nonzero. In fact we will show that both sides are nonnegative at all $\mu \in F$. In the left hand side we have

by (5.3) and (3.6),

$$\begin{aligned} q_{\underline{i}}(H) &= \prod_{j=1}^{\ell} q_{i_j}(H + \alpha_{i_1} + \cdots + \alpha_{i_{j-1}} + \frac{\alpha_{i_j}}{2}) \\ &= \exp \left(2\pi i \sum_{j=1}^{\ell} l_{i_j}(H + \alpha_{i_1} + \cdots + \alpha_{i_{j-1}} + \frac{\alpha_{i_j}}{2})/4 \right) \\ &\quad \cdot \prod_{j=1}^{\ell} |p_{i_j}(H + \alpha_{i_1} + \cdots + \alpha_{i_{j-1}} + \frac{\alpha_{i_j}}{2})|^{1/2} \end{aligned}$$

Setting $H = \mu$ in the exponential expression we get after dividing by $2\pi i/2$

$$\frac{1}{2} \sum_{j=1}^{\ell} l_{i_j} \left(\mu + \alpha_{i_1} + \cdots + \alpha_{i_{j-1}} + \frac{\alpha_{i_j}}{2} \right).$$

If $\mu = \alpha_{k_1} + \cdots + \alpha_{k_r}$ for some $\underline{k} = k_1 k_2 \cdots k_r \in \text{Seq}_2$, then using (4.10) this equals

$$\omega(\mu + \alpha_{i_1} + \cdots + \alpha_{i_{\ell}}) - \omega(\mu) = 0$$

since $\sum_{j=1}^{\ell} \alpha_{i_j} = m\alpha + n\beta = 0$. A similar argument can be made in the case of $\mu = -(\alpha_{k_1} + \cdots + \alpha_{k_r})$. This proves that

$$q_{\underline{i}}(\mu) \geq 0 \quad \text{for all } \mu \in F. \quad (5.8)$$

Next we prove that the same is true for the product in the right hand side of (5.4). Assuming μ is not a zero we have

$$\text{sgn} \left(\prod_{\lambda \in F} (\mu - \lambda)^{\text{ord}(\underline{i}, \lambda)} \right) = \text{sgn} \left(\prod_{\lambda \in F, \lambda > \mu} (\mu - \lambda)^{\text{ord}(\underline{i}, \lambda)} \right) = (-1)^{\sum_{\lambda \in F, \lambda > \mu} \text{ord}(\underline{i}, \lambda)}. \quad (5.9)$$

Using Lemma 5.3 we get

$$\sum_{\substack{\lambda \in F \\ \lambda > \mu}} \text{ord}(\underline{i}, \mu) = \sum_{\substack{\lambda \in F \\ \lambda > \mu}} \frac{1}{2} \sum_{j=1}^{\ell} \mathcal{L}_{i_j} \left(\lambda + \alpha_{i_1} + \cdots + \alpha_{i_{j-1}} + \frac{\alpha_{i_j}}{2} \right).$$

Interchanging the order of summation and making the change of variables $e = \lambda + \alpha_{i_1} + \cdots + \alpha_{i_{j-1}} + \frac{\alpha_{i_j}}{2} \in E_{i_j}$ we obtain

$$\frac{1}{2} \sum_{j=1}^{\ell} \sum_{\substack{e \in E_{i_j} \\ e > \mu + \alpha_{i_1} + \cdots + \alpha_{i_{j-1}} + \frac{\alpha_{i_j}}{2}}} \mathcal{L}_{i_j}(e).$$

Now use the definition (3.1) of l_j to get

$$\frac{1}{2} \sum_{j=1}^{\ell} l_{i_j} \left(\mu + \alpha_{i_1} + \cdots + \alpha_{i_{j-1}} + \frac{\alpha_{i_j}}{2} \right)$$

which as shown above equals zero. This proves that for any $\underline{i} \in \text{Seq}_2(m, n)$,

$$\sum_{\substack{\lambda \in F \\ \lambda > \mu}} \text{ord}(\underline{i}, \lambda) = 0 \quad \text{for all } \mu \in F \text{ such that } \text{ord}(\underline{i}, \mu) = 0 \quad (5.10)$$

and hence by (5.9),

$$\prod_{\lambda \in F} (\mu - \lambda)^{\text{ord}(\underline{i}, \lambda)} \geq 0 \quad \text{for all } \mu \in F. \quad (5.11)$$

This finishes the proof of the identity (5.4). \square

We now prove that C given in (5.1) is unitary.

Proposition 5.5. *The Casimir element for $\mathcal{A}(\mathcal{L})$ is unitary with respect to $*$. That is:*

$$C^* \cdot C = 1 = C \cdot C^* \quad (5.12)$$

Proof. Since $H^* = H$ and $\lambda \in F = \mathbb{Z}$ we have

$$C^* C = \prod_{\lambda \in F} (H - \lambda)^{-\text{ord}(\underline{i}, \lambda)} X(\underline{i})^* X(\underline{i}) \prod_{\lambda \in F} (H - \lambda)^{-\text{ord}(\underline{i}, \lambda)} = \prod_{\lambda \in F} (H - \lambda)^{-2 \text{ord}(\underline{i}, \lambda)} X(\underline{i})^* X(\underline{i})$$

By a straightforward calculation (see e.g. proof of [3, Lem. 5.5])

$$X(\underline{i})^* X(\underline{i}) = p_{\underline{i}}(H)$$

where $p_{\underline{i}}(H)$ is as in (5.3) with $p_i = P_i^{\mathcal{L}}$. By Lemma 5.4,

$$p_{\underline{i}}(H) = (q_{\underline{i}}(H))^2 = \prod_{\lambda \in F} (H - \lambda)^{2 \text{ord}(\underline{i}, \lambda)}.$$

This finishes the proof. \square

5.2. Pseudo-unitarizability of simple integral weight modules. We recall the classification of simple integral weight $\mathcal{A}(\mathcal{L})$ -modules from [3].

Consider the space $\mathbb{T}_{m,n} = \mathbb{R}^2 / \langle (m, n) \rangle$ equipped with the quotient topology. This space is homeomorphic to a doubly infinite cylinder. Let $\overline{\mathcal{L}} \subseteq \mathbb{T}_{m,n}$ be the configuration \mathcal{L} regarded as a union of closed line segments. Let $\overline{F} \subseteq \mathbb{T}_{m,n}$ be the image of \mathbb{Z}^2 under the canonical projection $\mathbb{R}^2 \rightarrow \mathbb{T}_{m,n}$.

Theorem 5.6 ([3, Thm. B]). (a) *There is a bijective correspondence between the set of isoclasses of simple integral weight $\mathcal{A}(\mathcal{L})$ -modules, and the set of pairs (D, ξ) where D is a connected component of $\mathbb{T}_{m,n} \setminus \overline{\mathcal{L}}$ and $\xi \in \mathbb{C}$ with $\xi = 0$ iff D is contractible.*
 (b) *Let $M(D, \xi)$ be the module corresponding to (D, ξ) . Each nonzero weight space $M(D, \xi)_{\lambda}$ is one-dimensional and*

$$\text{Supp}(M(D, \xi)) = \{x_1 \alpha_1 + x_2 \alpha_2 \mid (x_1, x_2) + \langle (m, n) \rangle \in \overline{F} \cap D\}.$$

- (c) For any incontractible D and $\xi \in \mathbb{C}^\times$ the action of the Casimir element C for $\mathcal{A}(\mathcal{L})$ from (5.1) is well-defined on $M(D, \xi)$ and $C|_{M(D, \xi)} = \xi \text{Id}_{M(D, \xi)}$.

The following lemma is immediate because the support of an integral weight module is contained in F which is a subset of \mathbb{R} .

Lemma 5.7. *If M is a simple integral weight $\mathcal{A}(\mathcal{L})$ -module then $\text{Supp}(M) = \text{Supp}(M^\#)$.*

Using the unitarity of the Casimir C from Proposition 5.5, we obtain the following description of the pseudo-unitarizable simple integral weight $\mathcal{A}(\mathcal{L})$ -modules.

Theorem 5.8. *Let D be a connected component of $\mathbb{T}_{m,n} \setminus \overline{\mathcal{L}}$.*

- (i) *If D is contractible, then $M(D, 0)$ is pseudo-unitarizable.*
- (ii) *If D is incontractible then $M(D, \xi)$ is pseudo-unitarizable if and only if $|\xi| = 1$.*

Proof. By Theorem 4.4, a simple weight module M is pseudo-unitarizable if and only if $M^\# \simeq M$.

(i) Put $M = M(D, 0)$. By Theorem 5.6, since $\text{Supp}(M^\#) = \text{Supp}(M) = D$ which is contractible, it follows that $M^\# \simeq M$ hence M is pseudo-unitarizable.

(ii) By Theorem 5.6 and Lemma 5.7, for any $\xi \in \mathbb{C}^\times$ there exists $\xi^\# \in \mathbb{C}^\times$ such that $M(D, \xi)^\# \simeq M(D, \xi^\#)$. Recall that ξ has the interpretation as being the eigenvalue of C . By Proposition 5.5, $C^* = C^{-1}$ and thus for any $f \in M^\#$ and $v \in M$,

$$(Cf)(v) = f(C^*v) = f(\xi^{-1}v) = (\bar{\xi}^{-1}f)(v).$$

This proves that $\xi^\# = \bar{\xi}^{-1}$. By the classification theorem again, $M(D, \xi) \simeq M(D, \bar{\xi}^{-1})$ if and only if $\xi = \bar{\xi}^{-1}$ or equivalently, $|\xi| = 1$. \square

5.3. Decomposition of $\Gamma_0(L_\xi)$ into irreducibles. Put

$$M_0 = \bigoplus_D M(D, 0) \quad M_\xi = \bigoplus_{D'} M(D', \xi) \quad (5.13)$$

where D (respectively D') runs over the set of contractible (respectively incontractible) connected components of $\mathbb{T}_{m,n} \setminus \overline{\mathcal{L}}$, and $\xi \in \mathbb{C}^\times$ is fixed.

Proposition 5.9. *For any $\xi \in \mathbb{C}^\times$ there is an isomorphism of $\mathcal{A}(\mathcal{L})$ -modules*

$$\Gamma_0(L_\xi) = M_0 \oplus M_\xi. \quad (5.14)$$

Proof. Each H -weight space of $\Gamma_0(L_\xi)$ is one-dimensional, spanned by f_k , $k \in \mathbb{Z}$. For each connected component D of $\mathbb{T}_{m,n} \setminus \overline{\mathcal{L}}$, there is a submodule of $\Gamma_0(L_\xi)$ whose support is exactly D . By the characterizing properties of the simple integral weight modules from Theorem 5.6, it remains to prove that if D is an incontractible component and f_k is one of the basis vectors where $k \in F(D)$, then $\Delta_\xi(C)f_k = \xi f_k$.

Let $\underline{i} \in \text{Seq}_2(m, n)$. We have

$$\begin{aligned} (\Delta_\xi(X(\underline{i}))f)(x, y) &= (\Delta_\xi(X_{i_\ell} \cdots X_{i_1})f)(x, y) \\ &= q_{i_\ell}(x\alpha + y\beta - \frac{\alpha_{i_\ell}}{2})q_{i_{\ell-1}}(x\alpha + y\beta - \alpha_{i_\ell} - \frac{\alpha_{i_{\ell-1}}}{2}) \cdots \\ &\quad \cdots q_{i_1}(x\alpha + y\beta - \alpha_{i_\ell} - \alpha_{i_{\ell-1}} - \cdots - \alpha_{i_2} - \frac{\alpha_{i_1}}{2}) \cdot f((x, y) - \mathbf{e}_{i_\ell} - \cdots - \mathbf{e}_{i_1}) \\ &= (q_{\underline{i}}(\tilde{H})f)(x - m, y - n) = \xi q_{\underline{i}}(\tilde{H}) \cdot f(x, y) \end{aligned}$$

We have shown that as operators on $\Gamma_0(L_\xi)$ we have

$$\Delta_\xi(X(\underline{i})) = \xi q_{\underline{i}}(\tilde{H}) \quad (5.15)$$

where $\tilde{H} = \Delta_\xi(H)$. Consider the centralizing element $C \in \mathcal{A}(\mathcal{L})_{\text{loc}}$ given by (5.1). We have

$$\Delta_\xi(C) = \Delta_\xi \left(X(\underline{i}) \prod_{\lambda \in F} (H - \lambda)^{-\text{ord}(\underline{i}, \lambda)} \right) = \xi q_{\underline{i}}(\tilde{H}) \prod_{\lambda \in F} (\tilde{H} - \lambda)^{-\text{ord}(\underline{i}, \lambda)} = \xi \quad (5.16)$$

where we used Lemma 5.4 in the last step. We abused notation by applying Δ_ξ to an element of the localization, but the resulting operator is well-defined on any f_λ for λ in an incontractible component. This finishes the proof. \square

6. ON THE SIGNATURE OF THE INVARIANT INNER PRODUCT ON $M(D, \xi)$ AND INTERNAL EIGHT-VERTEX CONFIGURATIONS

6.1. Unitarizable simple integral weight $\mathcal{A}(\mathcal{L})$ -modules. By Theorem 5.6 each simple integral weight $\mathcal{A}(\mathcal{L})$ -module M is isomorphic to $M(D, 0)$ with D contractible or $M(D, \xi)$ with D incontractible and $\xi \in \mathbb{C}^\times$. As we saw in Theorem 5.8, the former are always pseudo-unitarizable and the latter iff $|\xi| = 1$. In this case there is a unique up to nonzero real multiple admissible form $\langle \cdot, \cdot \rangle$ on M by Theorem 4.4. Thus, identifying M with a submodule of $\Gamma_0(L_\xi)$ as in Proposition 5.9, we may define the *signature* of M to be

$$\sigma(M) = \{\dim M^+, \dim M^-\} \quad (6.1)$$

where M^\pm are the ± 1 eigenspaces of the fundamental symmetry J from Remark 4.8. All this just amounts to the following formula

$$\sigma(M) = \{s_+, s_-\} \quad s_\pm = \#\{\lambda \in F \mid \pm \langle v, v \rangle \geq 0 \text{ for all } v \in M_\lambda\}. \quad (6.2)$$

for some choice of invariant inner product $\langle \cdot, \cdot \rangle$. Changing the form to $-\langle \cdot, \cdot \rangle$ does not change $\sigma(M)$ as a set. Thus $\sigma(M)$ depends only on M and not on the choice of invariant inner product on M . We say that M is *definite* if $0 \in \sigma(M)$. Thus M is definite if and only if M is unitarizable.

Lemma 6.1. *The sign of the quadratic form $v \mapsto \langle v, v \rangle$ are the same on two adjacent weight spaces of weights λ and $\lambda + \alpha_i$ if and only if $P_i^{\mathcal{L}}(e) > 0$ where $e = \lambda + \alpha_i/2$ is the (midpoint of the) edge separating the weight spaces.*

$$\begin{array}{c} \times \\ \lambda \end{array} \quad \left| \quad \begin{array}{c} \times \\ \lambda + \alpha_1 \end{array} \right.$$

FIGURE 2. Adjacent weight spaces in the support of a simple integral weight module.

Proof. Suppose that $v_\lambda \in V$ is a nonzero eigenvector of H with eigenvalue λ . Then

$$\langle X_i^+ v_\lambda, X_i^+ v_\lambda \rangle = \langle X_i^- X_i^+ v_\lambda, v_\lambda \rangle = \langle P_i^{\mathcal{L}}(H + \frac{\alpha_i}{2})v_\lambda, v_\lambda \rangle = P_i^{\mathcal{L}}(\lambda + \frac{\alpha_i}{2})\langle v_\lambda, v_\lambda \rangle. \quad (6.3)$$

□

Recall the cylinder $\mathbb{T}_{m,n} = \mathbb{R}^2 / \langle (m, n) \rangle$. For any subset $D \subseteq \mathbb{T}_{m,n}$ we put

$$E_i(D) = \{x\alpha + y\beta \in E_i \mid (x, y) \in (\mathbb{Z}^2 + \frac{1}{2}\mathbf{e}_i) \cap D\}. \quad (6.4)$$

where $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$. Thus $E_1(D)$ (respectively $E_2(D)$) is the set of vertical (respectively horizontal) edges that when drawn in a fundamental domain in \mathbb{R}^2 have their midpoint inside D . Similarly we put

$$\begin{aligned} F(D) &= \{x\alpha + y\beta \in F \mid (x, y) \in \mathbb{Z}^2 \cap D\}, \\ V(D) &= \{x\alpha + y\beta \in V \mid (x, y) \in (\mathbb{Z}^2 + \frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2)) \cap D\}. \end{aligned}$$

We call elements of these sets *internal edges*, *faces* and *vertices* in D .

The following gives a characterization of unitarizable simple integral weight $\mathcal{A}(\mathcal{L})$ -modules.

Proposition 6.2. *Let $M(D, \xi)$ be a pseudo-unitarizable simple integral weight $\mathcal{A}(\mathcal{L})$ -module. Then the following statements are equivalent.*

- (i) $M(D, \xi)$ is unitarizable.
- (ii) $w|_{F(D)}$ is constant where w was defined in (4.7), (4.10).
- (iii) $P_i^{\mathcal{L}}(e_i) > 0$ for all $e_i \in E_i(D)$ and $i \in \{1, 2\}$.
- (iv) $l_i(e_i) \in 2\mathbb{N}$ for all $e_i \in E_i(D)$ and $i \in \{1, 2\}$, where l_i was defined in (3.1).
- (v) At each internal vertical (respectively horizontal) edge e in D , there are an even number, counted with multiplicity, of vertical (respectively horizontal) edges occurring in \mathcal{L} whose midpoints are above the line through the midpoint of e of slope n/m .

Proof. (i) \Leftrightarrow (ii) was noted above. (ii) \Leftrightarrow (iii) follows from Lemma 6.1. (iii) \Leftrightarrow (iv) is immediate by (3.2). Lastly (iv) \Leftrightarrow (v) follows from Remark 3.3. □

6.2. Internal eight-vertex configurations and the signature of $M(D, \xi)$. We turn to the final problem of calculating the signature of $M(D, \xi)$ as defined in the previous subsection. Define the *internal vertex configuration* in D to be $\mathcal{L}^D = (\mathcal{L}_1^D, \mathcal{L}_2^D)$ where $\mathcal{L}_i^D : E_i(D) \rightarrow \{1, -1\}$ is given by

$$\mathcal{L}_i^D(e) = \text{sgn } P_i^{\mathcal{L}}(e) \quad \text{for } e \in E_i \text{ and } i = 1, 2. \quad (6.5)$$

Note that $P_i^{\mathcal{L}}(e) \neq 0$ at every $e \in E_i(D)$. We interpret the value $+1$ as the edge being absent in the configuration \mathcal{L}^D , and -1 as an edge present of multiplicity one. In figures edges in \mathcal{L}^D will be drawn dashed in red. We make the following observation.

Lemma 6.3. *Let D be any connected component of $\mathbb{T}_{m,n} \setminus \overline{\mathcal{L}}$. Then \mathcal{L}^D is an eight-vertex configuration in D . That is, at each internal vertex $v \in V(D)$, there are exactly eight possible local configurations, see Figure 3.*

Proof. At an internal vertex v , both sides of

$$P_1^{\mathcal{L}}(v + \alpha_2/2)P_2^{\mathcal{L}}(v + \alpha_1/2) = P_1^{\mathcal{L}}(v - \alpha_2/2)P_2^{\mathcal{L}}(v - \alpha_1/2)$$

are nonzero. Taking signs on both sides the claim follows. \square

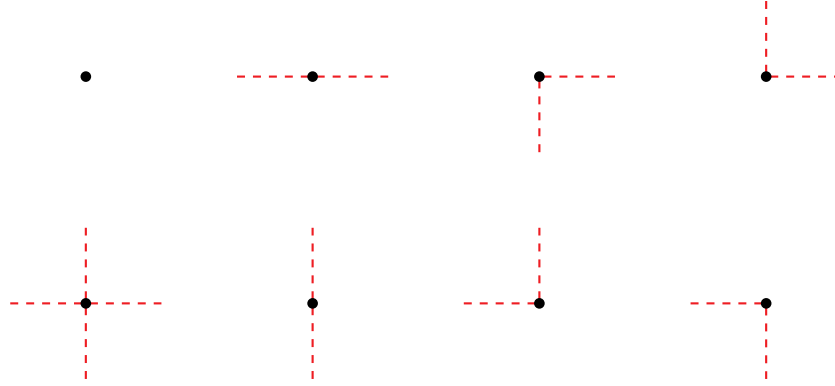


FIGURE 3. Local eight-vertex configurations.

We now describe an algorithm for determining the signature of a simple integral weight module $M(D, \xi)$ over a noncommutative Kleinian fiber product $\mathcal{A}(\mathcal{L})$.

Consider internal edges in D . For each vertical edge $e \in E_1(D)$, draw a straight line through the midpoint of the edge such that the line has slope n/m . Then count the number (with multiplicity) of vertical edges in \mathcal{L} whose midpoint is above that line. If that number is odd, color the edge e red. Otherwise leave it transparent. Repeat that for each vertical edge in D . Then carry out the analogous procedure for horizontal edges in D . After all internal edges of D have been either colored red or left transparent, the red edges will form the eight-vertex configuration \mathcal{L}^D . Then, by removing the union $\overline{\mathcal{L}^D}$ of line segments corresponding to red edges in D , this breaks D further into subcomponents $D^{(j)}$:

$$D \setminus \overline{\mathcal{L}^D} = \bigsqcup_j D^{(j)} \quad (6.6)$$

where \bigsqcup means disjoint union. On each sum of weight spaces

$$M(D, \xi)^{(j)} = \bigoplus_{\lambda \in F(D^{(j)})} M(D, \xi)_\lambda$$

the invariant inner product is positive or negative definite, by Lemma 6.1. Moreover if two connected components $D^{(j)}$ and $D^{(j')}$ are adjacent then the invariant inner product is positive definite on one of them and negative definite on the other. Thus by two-coloring the decomposition (6.6) of D and then counting the number c_i of connected components $D^{(j)}$ of each color $i \in \{+, -\}$, that gives the signature of $M(D, \xi)$:

$$\sigma(M(D, \xi)) = \{c_+, c_-\}. \quad (6.7)$$

In particular the signature is independent of ξ when $|\xi| = 1$.

7. EXAMPLES

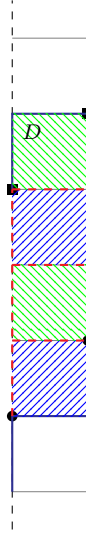


FIGURE 4. A $(1, 1)$ -periodic six-vertex configuration with two paths.

Example 7.1. Figure 4 shows a fundamental domain for $\mathbb{T}_{1,1} = \mathbb{R}^2 / \langle (1, 1) \rangle$ with the blue solid edges constituting the $d = 4$ case of the $(1, 1)$ -periodic configuration \mathcal{L} with

$$P_1^{\mathcal{L}}(u) = P_2^{\mathcal{L}}(u) = \left(u - \frac{1}{2}\right)\left(u - \frac{1}{2} - d\right)$$

where d is a positive integer. It was shown in [3] that the corresponding noncommutative Kleinian fiber product $\mathcal{A}(\mathcal{L})$ is related to d -dimensional evaluation modules for the affine Lie algebra $A_1^{(1)}$ and to a finite W-algebra associated to \mathfrak{sl}_4 . $\mathbb{T}_{1,1} \setminus \overline{\mathcal{L}}$ consists of three connected components, one of which is finite and denoted D . Since D has the homotopy type of a circle, there is a one-parameter family of d -dimensional simple integral weight $\mathcal{A}(\mathcal{L})$ -modules $M(D, \xi)$. Coloring the internal edges as in the algorithm in Section 6.2, we obtain that all internal edges are red. Thus if $|\xi| = 1$ then $M(D, \xi)$ is pseudo-unitarizable

with signature

$$\sigma(M(D, \xi)) = \begin{cases} \{d/2, d/2\} & d \text{ even} \\ \{(d-1)/2, (d+1)/2\} & d \text{ odd} \end{cases}$$

If we instead of $*$ consider the “Chevalley” involution \dagger on $\mathcal{A}(\mathcal{L})$ given by $(X_i^\pm)^\dagger = -X_i^\mp$, $H^\dagger = H$, then this is equivalent to changing signs of the polynomials $P_i^\mathcal{L}(u)$, hence \mathcal{L}^D changes into $-\mathcal{L}^D$, meaning now all internal edges in D are transparent. This recovers the well-known unitarizability of these loop modules regarded as modules over the affine Lie algebra $A_1^{(1)}$.

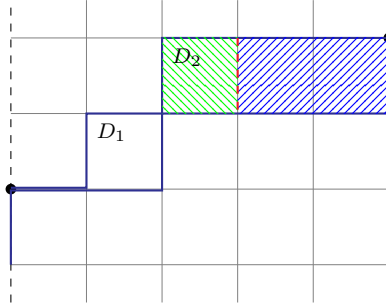


FIGURE 5. A fundamental domain for a $(5,2)$ -periodic configuration \mathcal{L} consisting of two vertex paths. $M(D_1, 0)$ is unitarizable, while $M(D_2, 0)$ has signature $\{1, 2\}$.

Example 7.2. Consider the $(5,2)$ -periodic higher spin vertex configuration \mathcal{L} in Figure 5 consisting of the two vertex lattice paths 1121112 and 1212111 (1 being a step right and 2 being a step up) with the same starting point. Removing these line segments from the doubly infinite cylinder $\mathbb{T}_{5,2} = \mathbb{R}^2 / \langle (5,2) \rangle$ leaves four connected components, two of which are finite, D_1 of area 1 and D_2 of area 3. Thus the noncommutative Kleinian fiber product $\mathcal{A}(\mathcal{L})$ has exactly two finite-dimensional simple modules, the one-dimensional $M(D_1, 0)$ and the three-dimensional $M(D_2, 0)$. The former is unitarizable (since there are no internal edges to check) while the latter has one red internal vertical edge, meaning an edge where $P_1(e) < 0$. By the algorithm in Section 6.2 this implies the signature of $M(D_2, 0)$ is $\{1, 2\}$.

Example 7.3. In Figure 6, $\mathcal{A}(\mathcal{L})$ has one 6-dimensional simple module $M(D_1, 0)$ and a one-parameter family of 14-dimensional simple modules $M(D_2, \xi)$, $\xi \in \mathbb{C}^\times$. Using the algorithm in Section 6.2 one checks that the module $M(D_1, 0)$ is unitarizable. For $|\xi| = 1$ the module $M(D_2, \xi)$ is pseudo-unitarizable of signature $\{7, 7\}$.

Example 7.4. Figure 7 shows a $(7,3)$ -periodic configuration \mathcal{L} such that the algebra $\mathcal{A}(\mathcal{L})$ has a unique finite-dimensional simple module $M(D, 0)$. This module has dimension 11 and signature $\{5, 6\}$.

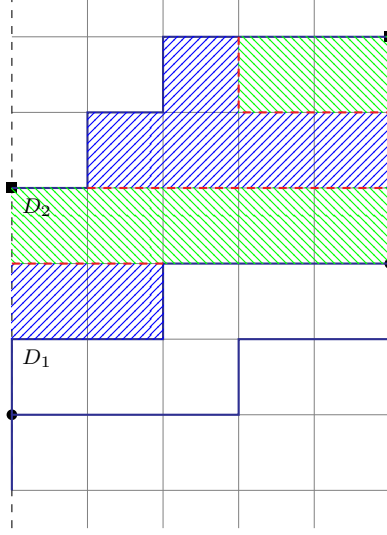


FIGURE 6. A $(5,2)$ -periodic example with \mathcal{L} consisting of three vertex paths. $M(D_1, 0)$ is definite, however $M(D_2, \xi)$ has signature $\{7, 7\}$ for each $\xi \in \mathbb{C}^\times$, $|\xi| = 1$.

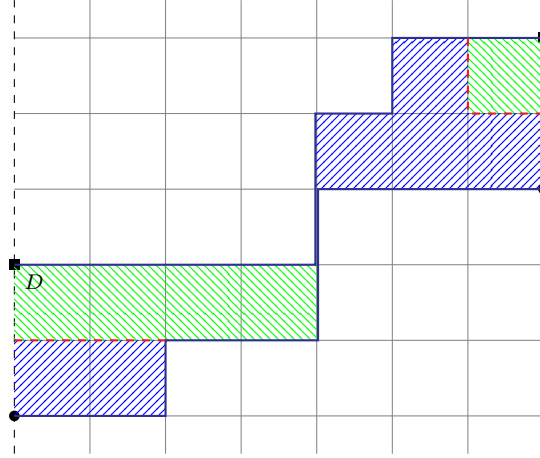


FIGURE 7. A $(7,3)$ -periodic configuration consisting of two paths. The signature of $M(D, 0)$ is $\{5, 6\}$.

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